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Solution of Some Type of Improper Fractional Integral

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus, we find the solution of some type of improper fractional integral. Differentiation under fractional integral sign and a new multiplication of fractional analytic functions play important roles in this paper. Moreover, some examples are given to illustrate our main result. In fact, our result is a generalization of ordinary calculus result.**

Keywords: **Jumarie type of R-L fractional calculus, improper fractional integral, differentiation under fractional integral sign, new multiplication, fractional analytic functions.**

I. INTRODUCTION

In recent decades, the applications of fractional calculus in various fields of science is growing rapidly, such as physics, biology, mechanics, electrical engineering, viscoelasticity, control theory, modelling, economics, etc [1-10]. However, the definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [11-15]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified R-L fractional calculus, we find the solution of the following improper α-fractional integral:

$$\left({}_{-\infty}I^{\alpha}_{+\infty}\right)\left[cos_\alpha(tx^\alpha)\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right], \tag{1}$$

where $0<\alpha\leq 1$, $(-1)^\alpha$ exists, and $t\geq 0$. Differentiation under fractional integral sign and a new multiplication of fractional analytic functions play important roles in this paper. On the other hand, some examples are provided to illustrate our main result. In fact, our result is a generalization of traditional calculus result.

II. PRELIMINARIES

At first, the fractional calculus used in this paper and its properties are introduced below.

Definition 2.1 ([16])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D^{\alpha}_{x}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^\alpha}dt\ . \tag{2}$$

And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{x_0}I^{\alpha}_{x}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{3}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([17])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{5}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([18])**:** Assume that x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([19])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{6}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{7}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ & = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{8}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ & = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{9}$$

Definition 2.5 ([20])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{10}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{11}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{12}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes k}. \tag{13}$$

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Definition 2.6 ([21])**:** If $0 < \alpha \leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{14}$$

In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2k}, \tag{15}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(2k+1)}. \tag{16}$$

Definition 2.7: Assume that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes -1}$.

Definition 2.8: The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

Theorem 2.9 (differentiation under fractional integral sign) ([22])**:** *Assume that* $0 < \alpha \leq 1$, *t is a real variable, and* $f_\alpha(x^\alpha)$ *is a α-fractional analytic function at* $x = 0$, *then*

$$\frac{d}{dt}\left({}_0I_x^\alpha\right)[f_\alpha(tx^\alpha)] = \left({}_0I_x^\alpha\right)\left[\frac{d}{dt}f_\alpha(tx^\alpha)\right]. \tag{17}$$

Theorem 2.10 ([22])**:** *Let* $0 < \alpha \leq 1$, *then the improper α-fractional integral*

$$\left({}_0I_{+\infty}^\alpha\right)\left[sin_\alpha(x^\alpha) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right] = \frac{\pi}{2}. \tag{18}$$

III. MAIN RESULT AND EXAMPLES

In this section, we obtain some type of improper fractional integral. Moreover, some examples are given to illustrate our result. At first, we need a lemma.

Lemma 3.1: *Let* $0 < \alpha \leq 1$, *and* $t \geq 0$, *then the improper α-fractional integral*

$$\left({}_0I_{+\infty}^\alpha\right)\left[sin_\alpha(tx^\alpha)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right] = \frac{\pi}{2}. \tag{19}$$

Proof $\left({}_0I_{+\infty}^\alpha\right)\left[sin_\alpha(tx^\alpha)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right]$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[sin_\alpha(tx^\alpha)\otimes\left(t\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\otimes\left({}_0D_x^\alpha\right)\left[t\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[sin_\alpha(x^\alpha)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right]$$

$$= \frac{\pi}{2}.$$

Q.e.d.

Theorem 3.2: *If* $0 < \alpha \leq 1$, $(-1)^\alpha$ *exists, and* $t \geq 0$, *then the improper α-fractional integral*

$$\left({}_{-\infty}I_{+\infty}^\alpha\right)\left[cos_\alpha(tx^\alpha)\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right] = \left(\frac{T_\alpha}{4}-\frac{\pi}{2}\right)e^t + \left(\frac{T_\alpha}{4}+\frac{\pi}{2}\right)e^{-t}. \tag{20}$$

Proof Let $p(t) = \left({}_0I_{+\infty}^\alpha\right)\left[cos_\alpha(tx^\alpha)\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right]$, then by differentiation under fractional integral sign,

$$\frac{d}{dt}p(t) = \frac{d}{dt}\left[\left({}_0I_{+\infty}^\alpha\right)\left[cos_\alpha(tx^\alpha)\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right]\right]$$

$$= \left({}_0I^{\alpha}_{+\infty} \right)\left[\frac{d}{dt}cos_{\alpha}(tx^{\alpha})\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}\right]$$

$$= \left({}_0I^{\alpha}_{+\infty} \right)\left[-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes sin_{\alpha}(tx^{\alpha})\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}\right]$$

$$= -\left({}_0I^{\alpha}_{+\infty} \right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\otimes sin_{\alpha}(tx^{\alpha})\otimes\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right]\right)^{\otimes -1}\right]$$

$$= -\left({}_0I^{\alpha}_{+\infty} \right)\left[\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+1-1\right]\otimes sin_{\alpha}(tx^{\alpha})\otimes\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right]\right)^{\otimes -1}\right]$$

$$= -\left({}_0I^{\alpha}_{+\infty} \right)\left[sin_{\alpha}(tx^{\alpha})\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\right]+\left({}_0I^{\alpha}_{+\infty} \right)\left[sin_{\alpha}(tx^{\alpha})\otimes\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right]\right)^{\otimes -1}\right]$$

$$= -\frac{\pi}{2}+\left({}_0I^{\alpha}_{+\infty} \right)\left[sin_{\alpha}(tx^{\alpha})\otimes\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right]\right)^{\otimes -1}\right]. \quad \text{(by Lemma3.1)} \tag{21}$$

Using differentiation under fractional integral sign again yields

$$\frac{d^2}{dt^2}p(t)$$

$$= \frac{d}{dt}\left(-\frac{\pi}{2}+\left({}_0I^{\alpha}_{+\infty} \right)\left[sin_{\alpha}(tx^{\alpha})\otimes\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right]\right)^{\otimes -1}\right]\right)$$

$$= \left({}_0I^{\alpha}_{+\infty} \right)\left[\frac{d}{dt}\left(sin_{\alpha}(tx^{\alpha})\otimes\left(\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right]\right)^{\otimes -1}\right)\right]$$

$$= \left({}_0I^{\alpha}_{+\infty} \right)\left[cos_{\alpha}(tx^{\alpha})\otimes\left(\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right]\right)^{\otimes -1}\right]$$

$$= p(t). \tag{22}$$

Therefore,

$$p(t) = C_1e^t + C_2e^{-t}\,, \tag{23}$$

where C_1, C_2 are constants. Since

$$p(0)$$

$$= \left({}_0I^{\alpha}_{+\infty} \right)\left[\left(\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right]\right)^{\otimes -1}\right]$$

$$= arctan_{\alpha}(x^{\alpha})\Big|_{x=0}^{x=+\infty}$$

$$= \frac{T_{\alpha}}{4}. \tag{24}$$

It follows that

$$C_1 + C_2 = \frac{T_{\alpha}}{4}. \tag{25}$$

Furthermore,

$$\frac{d}{dt}p(t)\Big|_{t=0} = -\frac{\pi}{2}. \tag{26}$$

Hence,

$$C_1 - C_2 = -\frac{\pi}{2}. \tag{27}$$

So,

$$C_1 = \frac{T_\alpha}{8} - \frac{\pi}{4}, \tag{28}$$

$$C_2 = \frac{T_\alpha}{8} + \frac{\pi}{4}. \tag{29}$$

Therefore,

$$p(t) = \left(\frac{T_\alpha}{8} - \frac{\pi}{4}\right)e^t + \left(\frac{T_\alpha}{8} + \frac{\pi}{4}\right)e^{-t}. \tag{30}$$

Finally, we obtain

$$\left({}_{-\infty}I^{\alpha}_{+\infty}\right)\left[cos_\alpha(tx^\alpha)\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right]$$
$$= 2p(t)$$
$$= \left(\frac{T_\alpha}{4} - \frac{\pi}{2}\right)e^t + \left(\frac{T_\alpha}{4} + \frac{\pi}{2}\right)e^{-t}.$$

Q.e.d.

Example 3.3: *Suppose that* $0 < \alpha \le 1$, *and* $(-1)^\alpha$ *exists, then*

$$\left({}_{-\infty}I^{\alpha}_{+\infty}\right)\left[cos_\alpha(x^\alpha)\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right] = \left(\frac{T_\alpha}{4} - \frac{\pi}{2}\right)e + \left(\frac{T_\alpha}{4} + \frac{\pi}{2}\right)e^{-1}, \tag{31}$$

$$\left({}_{-\infty}I^{\alpha}_{+\infty}\right)\left[cos_\alpha(3x^\alpha)\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right] = \left(\frac{T_\alpha}{4} - \frac{\pi}{2}\right)e^3 + \left(\frac{T_\alpha}{4} + \frac{\pi}{2}\right)e^{-3}, \tag{32}$$

$$\left({}_{-\infty}I^{\alpha}_{+\infty}\right)\left[cos_\alpha(\sqrt{2}x^\alpha)\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right] = \left(\frac{T_\alpha}{4} - \frac{\pi}{2}\right)e^{\sqrt{2}} + \left(\frac{T_\alpha}{4} + \frac{\pi}{2}\right)e^{-\sqrt{2}}, \tag{33}$$

$$\left({}_{-\infty}I^{\alpha}_{+\infty}\right)\left[cos_\alpha\left(\frac{\sqrt{3}}{4}x^\alpha\right)\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right] = \left(\frac{T_\alpha}{4} - \frac{\pi}{2}\right)e^{\frac{\sqrt{3}}{4}} + \left(\frac{T_\alpha}{4} + \frac{\pi}{2}\right)e^{-\frac{\sqrt{3}}{4}}. \tag{34}$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus, we find the exact solution of some type of improper fractional integral. Differentiation under fractional integral sign and a new multiplication of fractional analytic functions play important roles in this article. On the other hand, we provide some examples to illustrate our main result. In fact, the major result we obtained is a natural generalization of the result in classical calculus. In the future, we will continue to use our methods to study the problems in engineering mathematics and fractional differential equations.

REFERENCES

[1] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp, 41-45, 2016.

[2] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[3] F. C. Meral, T. J. Royston, R. Magin, Fractional calculus in viscoelasticity: an experimental study, Communications in Nonlinear Science and Numerical Simulation, vol. 15, no. 4, pp. 939-945, 2010.

[4] R. L. Bagley and P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, Journal of Rheology, vol. 27, no. 3, pp. 201-210, 1983.

[5] R. L. Magin, Fractional calculus models of complex dynamics in biological tissues, Computers & Mathematics with Applications, vol. 59, no. 5, pp. 1586-1593, 2010.

[6] R. Caponetto, G. Dongola, L. Fortuna, I. Petras, Fractional order systems: modeling and control applications, Singapore: World Scientific, 2010.

[7] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[8] R. Hilfer (ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[9] V. V. Uchaikin, Fractional derivatives for physicists and engineers, vol. 1, Background and Theory, vol. 2, Application. Springer, 2013.

[10] M. Teodor, Atanacković, Stevan Pilipović, Bogoljub Stanković, Dušan Zorica, Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes, John Wiley & Sons, Inc., 2014.

[11] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[12] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, Inc., 1974.

[13] S. Das, Functional Fractional Calculus, 2nd ed. Springer-Verlag, 2011.

[14] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[15] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, USA, 1993.

[16] C. -H. Yu, Using trigonometric substitution method to solve some fractional integral problems, International Journal of Recent Research in Mathematics Computer Science and Information Technology, vol. 9, no. 1, pp. 10-15, 2022.

[17] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp. 32-38, 2015.

[18] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[19] C. -H. Yu, Study on some properties of fractional analytic function, International Journal of Mechanical and Industrial Technology, vol. 10, no. 1, pp. 31-35, 2022.

[20] C. -H. Yu, A Study of some fractional differential problems, International Journal of Mathematics and Physical Sciences Research, vol. 10, no. 1, pp. 52-58, 2022.

[21] C. -H. Yu, Exact solution of linear system of fractional differential equations with constant coefficients, International Journal of Mechanical and Industrial Technology, vol. 10, no. 2, pp. 1-7, 2022.

[22] C. -H. Yu, Application of differentiation under fractional integral sign, International Journal of Mathematics and Physical Sciences Research, vol. 10, no. 2, pp. 40-46, 2022.